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GRAPH PRODUCTS AND MONOCHROMATIC MULTIPLICITIES

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Arcane two-edge-colourings of complete graphs were described in [13], in which there are significantly fewer monochromatic K_r's than in a random colouring (so disproving a conjecture of Erdős [2]). Jagger, Šťovíček and Thomason [7] showed that the same colourings have fewer monochromatic G's than do random colourings for any graph G containing K_4.

The purpose of this note is to point out that these colourings are not as obscure as might appear. There is in fact a large, natural and easily described class of colourings of the above kind; the specific examples used in [13] and [7] fall into this class.

1. Introduction

Ramsey's theorem states that if the edges of a large complete graph are coloured with two colours then the graph must contain monochromatic copies of the complete graph K_r. It is natural to ask what is the minimum number of monochromatic K_r's in any such colouring. Erdős [2] suggested that the minimum might be achieved (more or less) by random colourings; specifically, that for large n the number of monochromatic K_r's in a two-colouring of K_n is at least $2^{1-\binom{r}{2}}\binom{n}{r}+o(n^r)$, this being the number in a random colouring. Whilst true for $r\le 3$ (see Goodman [5] and Lorden [8]), the conjecture was shown in [13] to be false for $r\ge 4$. The colouring used in [13] to beat random colourings is not at all one which would spring readily to mind; it is presented without motivation and gives the impression of having fallen from heaven. For the case $r=4$ other (more straightforward) examples have been found by Franek and Rödl [3].

The question of how to achieve the minimum monochromatic multiplicity can be asked of any graph G, not just of K_r, and Burr and Rosta [1] conjectured that random colourings would be best for every graph G. Sidorenko [11] and [12] proved this correct for trees and odd cycles but showed it to be false for the graph consisting of a triangle with a pendant edge. Jagger, Šťovíček and Thomason [7] examined the colourings used in [13] and showed that they improve on random colourings

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for any graph G containing K_4. Further information on the Burr-Rosta conjecture may be found in [7] and in Jagger [6].

Intuitively, it might seem very plausible that random colourings, with their uniform distribution of colours, will yield low numbers of monochromatic subgraphs. Colourings containing, say, large monochromatic complete subgraphs would appear unlikely to compete. However, now that it is known that random colourings are not optimal, this view must be discarded.

One of the simplest ways to define a graph (or, equivalently, a two-edge-colouring of K_n) is as a product of smaller graphs. For our purposes, the product of particular interest is the tensor product (defined below), because the number of monochromatic subgraphs in a tensor product of two 'factor' graphs is determined in a very straightforward way by the numbers in the two factors (Lemma 2, implicit in [7]). Since these numbers are easily computed for graphs with only a handful of vertices, it is no effort at all to investigate a large variety of different colourings which are products of small graphs.

Remarkably, it turns out that by this method it is almost impossible *not* to find a colouring which beats a random one. Indeed, factors with only three vertices will work, and these can easily be examined by hand. If computers are employed to widen the search, a wealth of counterexamples to Erdős' conjecture can be supplied, constrained only by the time available, which grows rapidly with both the order of the factor graph and the order of the graph G whose monochromatic copies are being counted. The method is therefore very handy for $G = K_4$ but gives no new information for large G. The most interesting new colourings produced are vertex-transitive $(n/2)$-regular colourings of K_n, in which the proportion of monochromatic K_4's is less than $1/33$ (compared with $1/32$ for a random colouring. Note that, by a theorem of Giraud [4], the proportion in such a graph cannot be less than $1/35$).

The investigation of various small graphs, described in this paper, was much more extensive than would otherwise have been possible, because of the availability of the excellent *nauty* program of Brendan McKay [9]. The use of *nauty* and various ancillary programs custom designed by McKay is gratefully acknowledged, as is the supply by McKay and Gordon Royle of a list of all Cayley graphs of orders up to 31 (see [10]).

The original colourings employed in [13] and [7] are also products of small graphs. Thus they will appear less unnatural in the light of what follows. But curiously, as we shall see, amidst the veritable plethora of new examples they remain the best.

2. Graph products

Without further ado, let us define the products in question. Given two graphs J_1 and J_2, their *tensor product* $J_1 \otimes J_2$ is a graph with vertex set $V(J_1 \otimes J_2) = V(J_1) \times V(J_2)$. The edges of $J_1 \otimes J_2$ are determined by $(v,w)(v',w') \in E(J_1 \otimes J_2)$ if *either* $vv' \in E(J_1)$ *or* $ww' \in E(J_2)$, *but not both.*

Note that, for a fixed vertex $w \in J_2$, the vertices $\{(v,w) : v \in J_1\}$ span a copy of J_1 in $J_1 \otimes J_2$. Moreover, if J_2 is the empty graph $\overline{K_m}$ of order m, then $J_1 \otimes \overline{K_m}$ is the usual m-fold cover of J_1. If we are prepared to regard isomorphic graphs as equal (and we are), it is easily checked that $J_1 \otimes J_2 = J_2 \otimes J_1$ and $(J_1 \otimes J_2) \otimes J_3 = J_1 \otimes (J_2 \otimes J_3)$.

As already mentioned, it is quite feasible to calculate the number of monochromatic K_4's in a tensor product. We will show that there are lots of ways to choose just a few small graphs whose product, when tensored with $\overline{K_m}$, will provide (as $m \to \infty$) a sequence of colourings with fewer monochromatic K_4's than random colourings. For example, $K_3^{\otimes 7}$ is one such product (meaning the product of seven copies of K_3).

The graphs used in [13] and [7] are defined as follows. Let V be a $2k$-dimensional vector space over $GF(2)$ with a specified basis. Define a quadratic form q on V by

$$q(x) = q((\xi_1, \ldots, \xi_{2k})) = \xi_1^2 + \xi_2^2 + \xi_1\xi_2 + \xi_3\xi_4 + \ldots + \xi_{2k-1}\xi_{2k}.$$

The graph T_k^- has vertex set V, and $xy \in E(T_k^-)$ if $q(x+y) = 1$. Now, the space V is the orthogonal sum of k subspaces of order 4, spanned by the $(2i-1)$'st and $(2i)$'th basis vectors, $1 \le i \le k$. Correspondingly, the graph T_k^- is the tensor product of k graphs of order 4; it can be checked that these graphs are one copy of K_4 and $k-1$ copies of M, where M is the graph of order 4 consisting of just two independent edges. That is to say, $T_k^- = K_4 \otimes M^{\otimes(k-1)}$. The factorization is not unique, since $M \otimes M = K_4 \otimes K_4$. It is shown in [13] that the colourings $T_r^- \otimes \overline{K_m}$ have few monochromatic K_r's.

3. Monochromatic multiplicities

This section is a summary of an elementary discussion in [7]. Let G be a graph of order r with $V(G) = \{1, \ldots, r\}$. Given a graph J, we associate with J the colouring of $K_{|J|}$ in which the edges of J are red and those of the complement, $\overline{J}$, are blue. A *homomorphic copy* of G in J is a map $f : V(G) \to V(J)$ such that $f(u)f(v) \in E(J)$ whenever $uv \in E(G)$. Let $\mathrm{Hom}(G;J)$ be the number of homomorphic copies G in J. The number of monochromatic subgraphs isomorphic to G in the colouring J is then equal to $(\mathrm{Hom}(G;J) + \mathrm{Hom}(G;\overline{J}))(1 + O(|J|^{-1}))$; this is because the number of monochromatic copies is of order $|J|^r$ (as follows easily from Ramsey's theorem) whereas the number of homomorphic copies which are not isomorphic is clearly at most $|J|^{r-1}$. Note here that we are counting what might be called 'labelled' copies of G in J; the number of 'unlabelled' copies is of course obtained by dividing by the order of the automorphism group of G.

Now associate with J the $|J| \times |J|$ matrix $A(J) = (a(u,v))_{u,v \in J}$, whose entries are indexed by the vertices of J and are defined by $a(u,v) = -1$ if $uv \in E(J)$

and $a(u,v) = 1$ otherwise. Note that the diagonal entries of $A(J)$ are all 1, and therefore $A(\overline{J}) \neq -A(J)$; this asymmetry will be referred to later. Consider now the expression

$$\sum_{u_1 \in J} \cdots \sum_{u_r \in J} \prod_{ij \in E(G)} (1 - a(u_i, u_j)) \quad + \quad \sum_{u_1 \in J} \cdots \sum_{u_r \in J} \prod_{ij \in E(G)} (1 + a(u_i, u_j)).$$

Clearly the first sum equals $2^{e(G)}$ times the number of monochromatic copies of G in J, whereas the second sum equals $2^{e(G)}\mathrm{Hom}(G; \overline{J})$ since we don't require that the vertices u_i be distinct. If we now expand the product in the first sum we get $2^{e(G)}$ sums, one sum for each spanning subgraph F of G, namely the sum $\pm|J|^r \Psi(J;F)$ where

$$\Psi(J;F) = \frac{1}{|J|^r} \sum_{u_1 \in J} \cdots \sum_{u_r \in J} \prod_{ij \in E(F)} a(u_i, u_j).$$

The sum $|J|^r \Psi(J;F)$ from the first expansion will be either doubled or cancelled by the corresponding sum from the second expansion, according as $e(F)$ is even or odd. These observations are summarized in the next lemma.

Lemma 1. *Let G and J be graphs, and for each spanning subgraph $F \subseteq G$, let $\Psi(J;F)$ be defined as above. Then the number of (labelled) monochromatic subgraphs in J isomorphic to G is given by $2^{1-e(G)}|J|^r(1+O(|J|^{-1}))\sigma(J;G)$, where*

$$\sigma(J;G) = \sum_{F \subseteq G} \Psi(J;F),$$

the sum being over all spanning subgraphs of G with an even number of edges.

Note that $2^{1-e(G)}|J|^r(1+O(|J|^{-1}))$ is the number of monochromatic copies of G contained in a random colouring J, so to find colourings which beat random colourings we must look for a sequence of large colourings J for which $\sigma(J;G) < 1$.

Here are a few remarks about the behaviour of $\Psi(J;F)$. Clearly $-1 < \Psi(J;F) \leq 1$ for all J and for all F. Moreover $\Psi(J;F) = 1$ if either J or F is the empty graph. More generally $\Psi(J; F \cup K_1) = \Psi(J;F)$, where $F \cup K_1$ denotes the addition of an isolated vertex to F. If F has only two edges then it is either the matching M or the path P of length 2. The value of $\Psi(J;M)$ is the square of the average value of the entries in $A(J)$. The value of $\Psi(J;P)$ is proportional to the sum of the squares of the row sums. In particular, both $\Psi(J;M)$ and $\Psi(J;P)$ are non-negative. The latter implies that $\sigma(J;K_3) \geq 1$ for all J, though this would follow also from Theorem 3 below and Goodman's theorem [5]. Finally, observe that if F is non-empty and J is random then $\Psi(J;F) \approx 0$.

4. Products and multiplicities

The crucial property of the function $\Psi(J;F)$, and the reason the matrix $A(J)$ was defined in the way it was, is that Ψ is a *multiplicative function* of J, in the following sense.

We defined earlier the tensor product of two graphs. The usual definition of the tensor product of the two square matrices $A=(a_{ij})_{i,j=1}^{n}$ and $B=(b_{kl})_{k,l=1}^{m}$ is the square matrix $A\otimes B$ of order nm, given by $(A\otimes B)_{(i,k)(j,l)}=a_{ij}b_{kl}$. This product is also known as the Kronecker product. It is easily seen that the matrix $A(J_1\otimes J_2)$ associated with the graph $J_1\otimes J_2$ is just the ordinary matrix tensor product $A(J_1)\otimes A(J_2)$; one reason for specifying that the diagonal entries of $A(J)$ equal one is to make this identity work.

The usefulness of tensor products in studying monochromatic multiplicities is given by the next observation, which we state as a lemma.

Lemma 2. *Let J_1, J_2 be graphs. Let F be graph of order r and let $\Psi(J;F)$ be as defined above. Then*

$$\Psi(J_1\otimes J_2;F)=\Psi(J_1;F)\Psi(J_2;F).$$

Proof. The verification of the claimed identity is just a trivial calculation. Let A, B and C be the matrices associated with $J_1\otimes J_2$, J_1 and J_2 respectively. Then

$$\begin{aligned}\Psi(J_1\otimes J_2;F)&=\frac{1}{|J_1\otimes J_2|^r}\sum_{u_1\in J_1\otimes J_2}\cdots\sum_{u_r\in J_1\otimes J_2}\prod_{ij\in E(F)}a(u_i,u_j)\\&=\frac{1}{|J_1|^r|J_2|^r}\sum_{v_1\in J_1,w_1\in J_2}\cdots\sum_{v_r\in J_1,w_r\in J_2}\prod_{ij\in E(F)}b(v_i,v_j)c(w_i,w_j)\\&=\Psi(J_1;F)\Psi(J_2;F),\end{aligned}$$

as asserted. ∎

One consequence of the above remarks can be stated as follows.

Theorem 3. *Let G and J be graphs such that $\sigma(J;G)<1$. Then the sequence of graphs $J\otimes\overline{K_n}$ provides a counterexample to the Burr-Rosta conjecture for G.*

Proof. $\Psi(\overline{K_n};F)=1$ for all F, and the conclusion follows from Lemmas 2 and 1. ∎

Recall that $J\otimes\overline{K_n}$ is just the n-fold cover of J. Theorem 3 is the basis for the counterexamples in [13] and [7]. However we can exploit Lemma 2 much further, as illustrated for example by the following result. Let $J^{\otimes k}$ be the tensor product of k copies of J.

Theorem 4. *Let G be a graph, and let J be a graph such that maximum value of $|\Psi(J;F)|$, over all non-empty spanning subgraphs $F \subseteq G$ of even size, is attained only by graphs with $\Psi(J;F) < 0$. Then $\sigma(J^{\otimes k};G) < 1$ for all sufficiently large odd k, and in particular the Burr-Rosta conjecture fails for G.*

Proof. From Lemma 2 we have that

$$\sigma(J^{\otimes k};G) = 1 + \sum_{\emptyset \neq F \subseteq G} \Psi(J;F)^k,$$

and the theorem follows at once from Theorem 3. ∎

5. Counterexamples for K_4

The observations of the preceding sections give rise at once to a very simple strategy for generating colourings with few monochromatic copies of some given graph G. For various small graphs J, it is a simple matter to compute directly the value of $\sigma(J;G)$. Even if we are unlucky at first, and don't find any J's with $\sigma(J;G) < 1$, we may find a J satisfying the conditions of Theorem 4. Moreover we can use Lemma 2 to compute the value of σ for various tensor products of different Js, which also might bring joy.

Let us examine the specific case $G = K_4$, which is the most interesting case to begin with. Now K_4 has 32 spanning subgraphs of even size, namely an empty graph, 12 paths P of length 2, 3 matchings M of size 2, 12 triangles with pendant edge T, 3 4-cycles C and one K_4. Thus, for any J, we have

$$\sigma(J^{\otimes k};K_4) = 1 + 12\Psi(J;P)^k + 3\Psi(J;M)^k + 12\Psi(J;T)^k + 3\Psi(J;C)^k + \Psi(J;K_4)^k.$$

It seems absurd to try a graph as small as $J = K_2$, and indeed it doesn't work. (It's not hard to see that $\Psi(K_2;F) = 0$ if F has a vertex of odd degree and $\Psi(K_2;F) = 1$ otherwise.) But our next choice $J = K_3$ *already works.* The values of $\Psi(K_3;F)$ for $F \subset K_4$ are shown in Table 1. It can be seen that the one with largest absolute value is $\Psi(K_3;K_4)$, which is negative. Hence, by Theorem 4, we know that $\sigma(K_3^{\otimes k};K_4) < 1$ if k is large. In fact it is easily verified that $\sigma(K_3^{\otimes 7};K_4) < 1$, so the graphs $K_3^{\otimes 7} \otimes \overline{K_m}$ give a counterexample to Erdős' conjecture.

J	P	M	T	C	K_4
K_3	0.1111	0.1111	−0.1852	0.4074	−0.4815
M	0.2500	0.2500	0.1250	0.2500	0.5000
K_4	0.2500	0.2500	−0.1250	0.2500	−0.5000
G_{18}	0.0000	0.0000	0.0000	0.1001	0.1975

Table 1. The values of $\Psi(J,F)$ for various graphs J and for $F \subseteq K_4$

The values of $\Psi(J,F)$ for $F \subseteq K_4$ are given in Table 1 for four graphs J, namely $J = K_3$, $J = M$, $J = K_4$ and $J = G_{18}$. The last graph will be described shortly. The graph used in [13] to disprove Erdős' conjecture for K_4 was a subgraph of $T_4^- = K_4 \otimes M^{\otimes 3}$, for which $\sigma = 0.9753$. This value is more or less matched by $K_3^{\otimes 2} \otimes K_4 \otimes M$; it can be reckoned from Table 1 that $\sigma(K_3^{\otimes 2} \otimes K_4 \otimes M; K_4) = 0.9783$.

All the facts stated so far are capable of being discovered, and can be verified, by hand calculation. However it is natural at this stage to begin a computer investigation of small graphs. It will then be found that there is no shortage of small graphs satisfying the conditions of Theorem 4 with $G = K_4$. Indeed, there are 10 graphs of order at most 6 and a further 174 graphs of order at most 8. In the search for other useful graphs, the value of $\Psi(J;F)$ for $F \subset K_4$ was found for a list of J's comprising all Cayley graphs of order at most 24 (this list, compiled using information from McKay and Royle as described in the introduction, includes nearly all vertex-transitive graphs). All products of up to 3 graphs from this list were tested. The more promising looking graphs were placed in a shortlist and all products of up to 6 of these graphs were tested. Similar lists were compiled containing all very small graphs, all moderately small regular graphs and all small edge-transitive graphs. Products of graphs from these lists were investigated as far as possible.

The upshot of all this electronic effort was that the smallest value of σ achieved was $\sigma(K_4 \otimes M \otimes G_{18}; K_4) = 0.9693 < 32/33$. The graph G_{18} is the complement of $K_3^{\otimes 2} \otimes \overline{K_2}$; it has order 18 and is 9-regular. Notice that the graphs $K_4 \otimes M \otimes G_{18} \otimes \overline{K_m}$ have order $n = 288m$, are vertex-transitive and are $n/2$ regular, making them the most 'symmetrical' as well as the best counterexamples to Erdős' conjecture to date. (The graph $K_4 \otimes M \otimes G_{18}$ is also minimal, in the sense that it is not possible to add or delete an edge without increasing σ.)

6. Counterexamples for other graphs

To compute $\sigma(J;K_r)$ for larger r, it is necessary to compute $\Psi(J;F)$ for all graphs F of order r and even size. This quickly becomes prohibitively time-consuming as r grows. However, for small r it is quite feasible. For $r = 5$ there are 18 graphs F which need checking, and for $r = 6$ there are 78. Lists of graphs J similar to those described above, but of orders up to 18, were checked. For $r = 5$ the example $\sigma(T_5^-; K_5) = 0.9056$ given in [13] was bettered somewhat by $\sigma(K_3 \otimes M^{\otimes 3}; K_5) = 0.8801$. For $r = 6$, no improvement was found on the example $\sigma(T_5^-; K_6) = 0.7641$.

It remains unexplained why the graph products $T_k^- = K_4 \otimes M^{\otimes(k-1)}$ perform so well even for those small values of r for which the performance of other products can be evaluated. It is unknown if other tensor products give colourings beating random colourings for K_r if r is large, though it seems likely that such products exist. The obvious theoretical advantage of the particular product T_k^- is that, because it can be represented in a different way (namely in terms of orthogonal geometry), the graph lends itself to a full analysis of its monochromatic subgraphs in a way that other graph products do not. In [7] it is shown that $\Psi(K_4;F) = (-1/2)^t$ and $\Psi(M;F) = (1/2)^t$, where $t \geq 1$ is an integer depending only on F, and that $t = 1$ only if F is a complete graph of even order (plus isolates). From this it follows at once from Theorem 3 that, for any graph G containing K_4, there is a value of k such that $\sigma(K_4 \otimes M^{\otimes k}; G) < 1$, so no graph containing K_4 can satisfy the Burr-Rosta conjecture.

It was pointed out in [7] that there are no 4-chromatic graphs for which the Burr-Rosta conjecture is known to hold. The smallest open case is $G = W_5$, the 5-spoked wheel. For each graph J for which $\sigma(J; K_6)$ was evaluated, the value of $\sigma(J; W_5)$ was also computed. No graphs were found with $\sigma(J; W_5) < 1$.

7. Further variations

Observe that the function $\Psi(J;F)$ is actually a function of a matrix rather than of a graph; in particular, Lemma 2 holds for products of arbitrary matrices. Two ways to extend the above constructions now immediately suggest themselves.

Consider first of all the fact that matrices which do not themselves represent graphs might still combine, when multiplied in a tensor product, to produce a matrix which does represent a graph. Any such product will be useful to us. However it turns out that no new examples can arise in this way. For the product matrix will represent a graph only if it is symmetric with ± 1 entries. Now the product matrix contains all possible products of entries from the individual factor matrices. It follows at once that any factor matrix must be symmetric with entries of the form $\pm\lambda$, for some λ. But then the same factor matrix divided by λ represents a graph (ignore for the time being the diagonal entries), and our graph turns out to be a product of graphs after all.

A second, more promising, extension is to drop the restriction that the diagonal entries of the matrices $A(J)$ be one. We can consider any symmetric matrix with ± 1 entries; such a matrix can be represented by a graph with loops at those vertices whose diagonal entry is -1, or alternatively, by a graph with an extra root vertex joined to those same vertices. The loops are ignored when equating a graph of order n with a colouring of K_n. The tensor product of graphs with loops is defined by the product of the associated matrices. Hence, for example, the tensor product of $\overline{K_m}$ and a graph with loops will be the m-fold cover of that graph but with those independent m-sets corresponding to vertices with loops replaced by K_m's.

Using programs of McKay as mentioned in the introduction, all graphs with loops of orders up to 6 were generated. There was no need to list a graph and its complement, since $A(\overline{J}) = -A(J)$ and a factor of -1 has no effect on $\Psi(J;F)$ if F has even size. Nevertheless the list had 2879 entries, and to extend it to order 7 would have added an extra 39632 graphs. For each graph in the list, the values of $\Psi(J;F)$ for $F \subseteq K_4$ were computed. Products of up to 3 of these graphs were tested, and larger products from the smaller list containing no graphs of orders 5 or 6. There was no evidence at all from these computations that any advantage was to be gained from the addition of loops.

8. A Final Improvement

The remark below is due entirely to a very thoughtful referee, who observed that, notwithstanding the second paragraph of the previous section, there is profit to be had from considering matrices with non-integer entries. Denote by M_n the set of symmetric $n \times n$ matrices with real entries from the interval $[-1,1]$ and let $I_n \subset M_n$ be those "graphic" matrices with ± 1 entries. The definitions of $A \otimes B$, $\Psi(A;F)$ and $\sigma(A;G)$ all carry over to $A \in M_n$. Then (as was remarked in [12])

$$\inf_n \min_{A \in M_n} \sigma(A;G) = \inf_n \min_{J \in I_n} \sigma(J;G).$$

To see this identity, let $A \in M_n$, let m be a large integer and let $B=(b(i,j))= A \otimes \overline{K_m}$. Now $\sigma(B;F)$ is very close to the expected value of $\sigma(J;F)$ of a random graphic matrix $J \in I_{mn}$ where $J(i,j) = \pm 1$ with probability $(1 \pm b(i,j))/2$; the difference lies only in those summands where some of $u_1,\dots,u_r$ coincide. Therefore, given $\epsilon > 0$ then if m is large enough there exists $J \in I_{mn}$ with $\sigma(J;F) < \sigma(B;F) + \epsilon = \sigma(A;F) + \epsilon$.

An equivalent and perhaps more explicit description of the colouring of K_{mn} associated with the matrix $A \otimes \overline{K_m}$ is as follows. The vertices of K_{mn} are partitioned into n disjoint subsets V_i, $1 \le i \le n$. Let $u \in V_i$ and $v \in V_j$. We then colour uv red with probability $(1-a(i,j))/2$. Thus the colouring is a random colouring. If $A \in I_n$ then this is just the usual m-fold cover of the graph A.

Here now is an example. Let $G_{18}(\epsilon)$ be the modification of G_{18} wherein the diagonal entries are $1-\epsilon$ instead of 1. Then it can be calculated that

$$\Psi(G_{18}(\epsilon);P) = \Psi(G_{18}(\epsilon);M) = 324\epsilon^2/18^4,\ \Psi(G_{18}(\epsilon);T) = 18^{-4}(-72\epsilon + 972\epsilon^2 - 54\epsilon^3 + 18\epsilon^4),$$
$$\Psi(G_{18}(\epsilon);C) = 18^{-4}(10512 - 288\epsilon + 1944\epsilon^2 - 72\epsilon^3 + 18\epsilon^4)$$

and

$$\Psi(G_{18}(\epsilon);K_4) = 18^{-4}(20736 + 972\epsilon^2 - 288\epsilon^3 + 270\epsilon^4 - 108\epsilon^5 + 18\epsilon^6).$$

Therefore

$$\sigma(K_4 \otimes M \otimes G_{18}(\epsilon);K_4) = \sigma(K_4 \otimes M \otimes G_{18};K_4) - \frac{1}{2592}\epsilon + O(\epsilon^2).$$

For small $\epsilon > 0$ this improves on the best construction given earlier.

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